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# External voltage sources and tunnelling in quantum wires

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## Abstract

We (re)consider in this paper the problem of tunnelling through an impurity in a quantum wire with arbitrary Luttinger interaction parameter. By combining the integrable approach developed in the case of quantum Hall edge states with the introduction of radiative boundary conditions to describe the adiabatic coupling to the reservoirs, we are able to obtain the exact equilibrium and non-equilibrium current. One of the most striking features observed is the appearance of negative differential conductances out of equilibrium in the strongly interacting regime  $g \leq 0.2$ . In spite of the various charging effects, a remarkable form of duality is still observed.

New results on the computation of transport properties in integrable impurity problems are gathered in the appendices. In particular, we prove that the TBA results satisfy a remarkable relation, originally derived using the Keldysh formalism, between the order  $T^2$  correction to the current out of equilibrium and the second derivative of this current at  $T = 0$  with respect to the voltage: this is the first direct link between the TBA approach out of equilibrium and the traditional Keldysh one

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## 1. Introduction

Tunnelling experiments [1] are one of the most efficient probes of the physics of Luttinger liquids, which is expected to describe the properties of one-dimensional conductors. The case of spinless Luttinger liquids has already been extensively studied, both theoretically and experimentally, in the context of edge states in a fractional quantum Hall bar where, in particular, shot noise measurements have led to the observation of fractional charge carriers [2]. The full crossover between the weak and strong back-scattering regimes has also been studied [3–5]: it exhibits in particular a duality between Laughlin quasiparticles

and electrons that is the result of the strong interactions in the system, and, ultimately, of integrability. From a theoretical point of view, it must be stressed that crossovers in this type of problem can only be properly studied with non-perturbative methods anyway. In fact, for the physics out of equilibrium, which plays a crucial role in the shot noise experiments for instance, numerical simulations do not even seem to be available.

Other one-dimensional conductors where Luttinger liquid physics could be observed include carbon nanotubes [6] or quantum wires in semiconductor heterostructures [7]. A key question for the latter examples is how to describe the application of an external voltage. In the fractional quantum Hall case, this turned out to be easy [5] because the left and right moving excitations are physically *separated* (the Luttinger liquid is really the ‘sum’ of two independent chiral ones), and put at a different chemical potential by the applied voltage. This will not be the case in a real quantum wire, where various charging effects have to be taken into account.

The matter led to some active debating [8, 9], and now seems quite settled. We follow here the approach of [9], which easily allows the inclusion of an impurity. We thus consider a gated quantum wire coupled adiabatically to 2D or 3D reservoirs. As in Landauer–Buttiker’s approach for non-interacting electrons [10, 11], these reservoirs are assumed to be ‘ideal’, and merely are there to inject bare densities of left and right movers in the wire. The interactions in the wire lead to the appearance of a non-trivial electrostatic potential and, in turn, to a renormalized charge density in the wire, in the absence of impurity. When the impurity is present, there is in addition a non-trivial spreading of the charges along the wire.

The key ingredient in the analysis of [9] is the equivalent of Poisson’s equation, which becomes a relation between the electrostatic potential  $\varphi$  and the charge density:  $e\varphi = u_0\rho$ . Here,  $u_0$  is related to the Luttinger liquid constant by  $g = (1 + u_0/\pi \hbar v_F)^{-1/2}$ . The electrostatic potential in turn shifts the band bottom, and thus the total density. There follows a relation between the bare injected densities and the true densities:

$$\begin{aligned}\rho_R^0 &= \frac{g^{-2} + 1}{2} \rho_R + \frac{g^{-2} - 1}{2} \rho_L \\ \rho_L^0 &= \frac{g^{-2} - 1}{2} \rho_R + \frac{g^{-2} + 1}{2} \rho_L.\end{aligned}\quad (1)$$

As for the bare densities themselves, they are related to the external voltage sources

$$\begin{aligned}\rho_R^0(-L/2) &= \frac{eU}{4\pi\hbar v_F} \\ \rho_L^0(L/2) &= -\frac{eU}{4\pi\hbar v_F}.\end{aligned}\quad (2)$$

The Hamiltonian, including the impurity term, is then, after bosonization,

$$H = \frac{\hbar v}{8\pi} \int dx [(\partial_x \phi_R)^2 + (\partial_x \phi_L)^2] + \lambda \cos[\sqrt{g}(\phi_R - \phi_L)] \quad (0) \quad (3)$$

where  $v = \frac{v_F}{g}$  is the sound velocity.

To proceed, one defines odd and even combinations of the bosonic field. Only the even field interacts with the outside potential and gives rise to a current. Setting

$$\phi_{e,o} = \frac{1}{\sqrt{2}} [\phi_R(x) \mp \phi_L(-x)] \quad (4)$$

the Hamiltonian of interest is

$$H_e = \frac{\hbar v}{8\pi} \int dx (\partial_x \phi_e)^2 + \lambda \cos(\sqrt{2g}\phi_e) \quad (0) \quad (5)$$

where  $\phi_e$  is a pure right moving field. In these new variables, the boundary conditions (2) are

$$(g^{-1} + 1)\rho_e(-L/2) - (g^{-1} - 1)\rho_e(L/2) = \sqrt{\frac{g}{2}} \frac{eU}{\pi\hbar v_F}. \tag{6}$$

In going from (1) to (6), the relation  $\rho_{R,L} = \frac{\sqrt{g}}{4\pi} \partial_x [\phi_R - \phi_L \pm \frac{1}{g} (\phi_R + \phi_L)]$ , which follows from bosonization, has been used. We have also defined  $\rho_{e,o} = \frac{1}{2\pi} \partial_x \phi_{e,o}$ . Finally, a mistake in [9] was corrected (see [12]).

Our goal is to compute the current  $I$  flowing through the system as a function of the applied voltage  $U$  in the limit of large wires<sup>3</sup>  $L \rightarrow \infty$ . In [9], this was accomplished in the case  $g = \frac{1}{2}$ , using a mapping on free fermions. In this paper, we shall solve the problem for general values of  $g$  using integrability of the boundary sine-Gordon model [13, 14]. This paper can be considered as a sequel—and to some extent a correction—to [5, 15], where the charging effects were not yet fully understood. It is also an extension of the short letter [16].

### 2. General formalism

First, we set  $e = v = \hbar = 1$  (so  $v_F = g$ ). To treat the interaction term at  $x = 0$  in an integrable way, one needs to choose an appropriate basis for the bulk, massless, right moving excitations, which obey  $e = p$ . For  $g$  generic, the basic excitations can be kinks or antikinks—carrying a  $\rho_e$  charge equal to  $\pm 1$ —and breathers. In the following we shall often restrict for technical simplicity to  $g = \frac{1}{t}$ ,  $t$  an integer. There are then  $t - 2$  breathers. We shall parametrize the energy of the excitations with rapidities  $e_j = m_j e^\theta$ . Here  $m_j$  is a parameter with the dimension of a mass; for kink and antikink,  $m_\pm = \mu$ , while for breathers,  $m_j = 2\mu \sin \frac{j\pi}{2(t-1)}$ ,  $j = 1, \dots, t - 2$ . The value of  $\mu$  is, of course, of no importance since the theory is massless, and in the following we shall simply set it equal to unity. The massless excitations enjoy factorized scattering in the bulk. At a temperature  $T$ , and with a choice of chemical potentials, they have densities given by solutions of the thermodynamic Bethe ansatz (TBA) equations, which we shall generically denote by  $\sigma_j$  (not to be confused with charge densities).

The key point is that these excitations also have a factorized scattering through the impurity, described by a transmission matrix  $T_{\pm\pm}$ . This matrix depends on the ratio of the energy of incident particles to a characteristic energy scale  $T_B$ . In the following, it is useful to parametrize  $T_B = e^{\theta_B}$ . The modulus squares of the transmission matrix have very simple expressions; we recall that

$$|T_{++}|^2 = \frac{e^{2((1/g)-1)(\theta-\theta_B)}}{1 + e^{2((1/g)-1)(\theta-\theta_B)}}. \tag{7}$$

Finally, we also recall how  $T_B$  is related to the bare coupling  $\lambda$  [5]:

$$T_B = (2 \sin \pi g)^{1/(1-g)} \frac{\Gamma(\frac{g}{2(1-g)})}{\sqrt{\pi} \Gamma(\frac{1}{2(1-g)})} [\lambda \Gamma(1 - g)/2]^{1/(1-g)}. \tag{8}$$

To proceed, we start by expressing the boundary conditions in terms of the massless scattering description:

$$\begin{aligned} \rho_e(L/2) &= \frac{1}{\sqrt{2g}} \int_{-\infty}^{\infty} (\sigma_+ |T_{++}|^2 + |T_{+-}|^2 \sigma_- - |T_{--}|^2 \sigma_- - |T_{+-}|^2 \sigma_+) d\theta \\ &= \frac{1}{\sqrt{2g}} \int_{-\infty}^{\infty} (\sigma_+ - \sigma_-) (|T_{++}|^2 - |T_{+-}|^2) d\theta. \end{aligned} \tag{9}$$

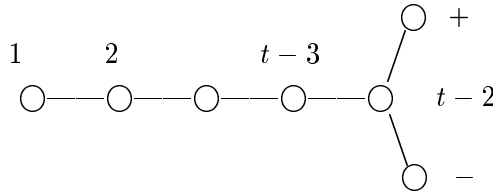
<sup>3</sup> In that limit, the position of the impurity along the wire is irrelevant, and is chosen at  $x = 0$  for convenience only.

Here,  $\sigma_{\pm}$  are the densities of kinks and antikinks; we set  $\sigma_{\pm} = n f_{\pm}$  where  $n = \sigma_+ + \sigma_-$  is the total density of states of kinks or antikinks and  $f_{\pm}$  are the filling fractions. The factor  $\frac{1}{\sqrt{2g}}$  occurs because  $\frac{1}{\sqrt{2g}}$  is the electric charge  $\frac{1}{2\pi} \int \partial_x \phi_e$  associated with the fundamental kinks of the problem.

All quantities of interest can be expressed using ‘pseudo energies’. The pseudo-energies for kinks and antikinks,  $\epsilon_{\pm}$ , are equal and are related to  $n$  (the total density of states of kinks or antikinks),  $n = \frac{1}{2\pi} \frac{d}{d\theta} \epsilon_{\pm}$ . Pseudo-energies for the other types of particles have also to be introduced, and they are obtained as solutions of the TBA system of equations

$$\epsilon_j = T \sum_k N_{jk} \frac{s}{2\pi} \star \ln(1 + e^{(\epsilon_k - \mu_k)/T}) \quad (10)$$

where  $s(\theta) = \frac{t-1}{\cosh((t-1)\theta)}$ ,  $g = \frac{1}{t}$ , and  $N_{jk}$  is the incidence matrix of the following TBA diagram:



Equations (10) have to be supplemented by asymptotic conditions  $\epsilon_j \approx m_j e^\theta$  as  $\theta \rightarrow \infty$ . In (10), the chemical potential vanishes for all the breathers which have no  $U(1)$  charge. For the kinks and antikinks,  $\mu_{\pm} = \mp \frac{W}{2}$ , where  $W$  has to be determined self-consistently. (The logic here is that the external potential and the temperature determine uniquely the average densities everywhere in the quantum wire. As always in macroscopic statistical mechanics, this can be described instead by a distribution with fixed chemical potentials, which is exactly what the TBA allows one to handle. By  $U(1)$  symmetry, it is known in advance that only the kinks and antikinks have a non-vanishing chemical potential  $\mu_{\pm} = \mp \frac{W}{2}$ .) The filling fractions are then

$$f_{\pm} = \frac{1}{1 + e^{(\epsilon_{\mp} W/2)/T}}. \quad (11)$$

The charge density on the left side of the impurity is simply  $\rho_e(-L/2) = \frac{1}{\sqrt{2g}} \int (\sigma_+ - \sigma_-) d\theta$  (it can be simply reexpressed in terms of  $W$ :  $\rho_e(-L/2) = \sqrt{\frac{g}{2}} \frac{W}{2\pi}$ ), so the boundary conditions equation (2) becomes

$$\int \left( |T_{++}|^2 + \frac{1}{g} |T_{+-}|^2 \right) (\sigma_+ - \sigma_-) d\theta = \frac{U}{2\pi}. \quad (12)$$

The other key equation in the solution of the problem follows from the charge density drop across the barrier

$$\Delta\rho = \rho(x < 0) - \rho(x > 0) = g \frac{V}{\pi}. \quad (13)$$

Here,  $\rho = \rho_R + \rho_L$  and  $V$  is the four terminal voltage (that is, the voltage difference measured by weakly coupled reservoirs on either side of the impurity; it consists of an electrostatic potential drop, plus an electrochemical contribution). By following the previous transformations, one finds that  $\Delta\rho = \Delta(\sigma_+ - \sigma_-)$ , and thus (13) is

$$\int |T_{+-}|^2 (\sigma_+ - \sigma_-) d\theta = g \frac{V}{2\pi}. \quad (14)$$

Finally, the tunnelling current  $I = \frac{U-V}{2\pi}$  is, from (13) and (12),

$$I = \int |T_{++}|^2 (\sigma_+ - \sigma_-) d\theta. \quad (15)$$

If  $T/T_B$  or  $U/T_B$  are large (the high energy, or weak back-scattering limit), the solution of (12) is  $\int (\sigma_+ - \sigma_-) d\theta \approx \frac{U}{2\pi}$ , and thus from (15),  $I \approx \int (\sigma_+ - \sigma_-) d\theta = \frac{U}{2\pi}$ . This, once physical units are reinstated, is  $I = \frac{e^2}{h} U$ , the expected formula for a spinless quantum wire.

From the foregoing system of equations, it is now easy to deduce the following identity giving the parameter  $W$  in terms of the physical voltage and current:

$$U = 2\pi \left(1 - \frac{1}{g}\right) I + W. \quad (16)$$

The following relation is also quite useful:

$$V = W - \frac{2\pi}{g} I. \quad (17)$$

### 3. Results

The limit  $g \rightarrow 1$ , which describes non-interacting electrons, is very simple. In that case indeed, the  $T$  matrix elements become rapidity independent, and the system of equations can readily be solved to give  $V = |T_{+-}|^2 U$ ,  $I = |T_{++}|^2 \frac{U}{2\pi}$ . Here, the transmission probability is not trivial in general since, as  $g \rightarrow 1$ ,  $\theta_B$  has to diverge to ensure a finite value of the bare coupling  $\lambda$  [14, 17].

The system of equations determining  $I$  can also be solved easily in the ‘classical limit’  $g \rightarrow 0$ , where [18] (this is detailed further in appendix A)

$$I \approx 2g \frac{T}{2\pi} \frac{\sinh(W/2T)}{I_{iW/2\pi T}(2x) I_{-iW/2\pi T}(2x)} \quad x = \frac{T_B}{4T} \quad (18)$$

where  $I$  are the usual Bessel functions and  $W$  follows from (16).

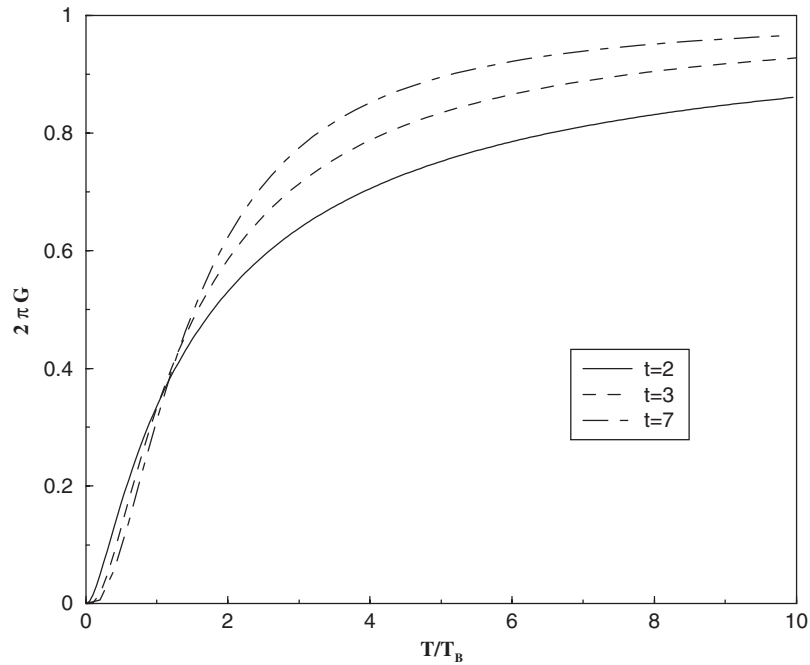
Closed form results can also be obtained for  $g = \frac{1}{2}$  (see below); besides, except at  $T = 0$ , one has to resort to a numerical solution of the TBA equations. To tackle the physics of this problem as  $g$  varies, we consider first the linear conductance at temperature  $T$ . In the limit  $U \rightarrow 0$ , the foregoing system of equations can easily be solved by linearization, giving rise to

$$G = \frac{1}{2\pi} \frac{\int |T_{++}|^2 \frac{d}{d\theta} \left( \frac{1}{1+e^{\epsilon/T}} \right) d\theta}{\int (|T_{++}|^2 + \frac{1}{g} |T_{+-}|^2) \frac{d}{d\theta} \left( \frac{1}{1+e^{\epsilon/T}} \right) d\theta} = \frac{1}{2\pi} \frac{G_0}{1 + (1 - \frac{1}{g}) G_0} \quad (19)$$

where  $G_0$  is the linear conductance in the quantum Hall effect problem [5] (the numerator of this equation). One of the roles of the denominator is to renormalize the conductance from  $g$  to unity in the UV region. In the case  $g = \frac{1}{2}$ , equation (19) can be evaluated in closed form to give

$$G = \frac{1}{2\pi} \frac{1 - \frac{T_B}{2\pi T} \Psi' \left( \frac{1}{2} + \frac{T_B}{2\pi T} \right)}{1 + \frac{T_B}{2\pi T} \Psi' \left( \frac{1}{2} + \frac{T_B}{2\pi T} \right)} \quad (20)$$

where  $\Psi$  is the digamma function. For values  $g = \frac{1}{t}$ ,  $t$  an integer,  $G$  is easily determined numerically by solving the system of TBA equations (10), and plotting the soliton pseudo-energy back into (19). Curves for various values of  $g$  are shown in figure 1; features entirely similar to those in [5] are observed, although all the curves now converge to the same value in the high temperature limit, in contrast with the quantum Hall edge case. The effect of the



**Figure 1.** We represent here the conductance as a function of the universal ratio of temperatures  $T/T_B$  for several values of  $g = 1/t$ ,  $t$  an integer. In this domain—which is the easiest to study numerically— $G$  has only a weak dependence on  $g$ . These curves interpolate between two limiting behaviours: for  $g = 1$ ,  $2\pi G$  should become a constant equal to  $1/2$ , while for  $g = 0$   $2\pi G$  should vanish for any finite value of  $T/T_B$ .

impurity is considerably amplified as  $g$  gets smaller, with  $G$  getting a discontinuity in the weak back-scattering limit as  $g \rightarrow 0$ . Indeed, letting  $U \rightarrow 0$  in (16), one finds

$$G = \frac{I}{U} \approx \frac{1}{2\pi} \frac{1}{\left(1 - \frac{1}{g}\right) + \frac{1}{g} I_0^2(2x)}. \quad (21)$$

As  $g \rightarrow 0$ ,  $G$  thus becomes a step function, jumping from  $\frac{1}{2\pi}$  to 0 as soon as  $T_B$  ( $T_B = 2\lambda$  for  $g = 0$ ) is turned on, for any temperature.

Another simple limit to study is the case  $T = 0$ , where results are far more intriguing. Consider first the classical limit, and think of  $I$  and  $U$  as functions of the parameter  $W$ : as  $W$  is swept, one finds that  $I$  vanishes while  $U$  increases up to  $\pi T_B$ , then goes back to zero, beyond which  $I$  increases like  $I = \frac{U}{2\pi}$ . In other words, the system behaves either like a perfect insulator, or like a perfect conductor! This very singular behaviour is the  $g \rightarrow 0$  limit of a multivalued  $I$ - $U$  characteristic with regions of negative differential conductance [16], that we now study in more detail.

Indeed, the TBA equations can be solved in closed form in the limit  $T \rightarrow 0$ . In that limit,  $\sigma_- = 0$ , the integrals run only from  $-\infty$  to a cut off (Fermi) rapidity  $A$ , and  $\sigma_+ = n$  follows from the solution of the integral equation

$$n(\theta) - \int_{-\infty}^A \Phi(\theta - \theta') n(\theta') d\theta' = \frac{e^\theta}{2\pi} \quad (22)$$

while  $A$  is determined by the condition that  $\epsilon_+^h(A) = 0$ , where

$$\epsilon_+^h(\theta) - \int_{-\infty}^A \Phi(\theta - \theta') \epsilon_+^h(\theta') d\theta' = \frac{W}{2} - e^\theta. \quad (23)$$

In that equation,  $\Phi$  is the derivative of the log of the kink  $S$  matrix

$$\Phi(\theta) = \int_{-\infty}^{\infty} e^{-i\omega\theta} \frac{\sinh \pi \left( \frac{2g-1}{2(1-g)} \right) \omega}{2 \cosh \frac{\pi\omega}{2} \sinh \frac{\pi g\omega}{2(1-g)}} \frac{d\omega}{2\pi}.$$

Since  $W$  determines  $A$  uniquely (one finds  $e^A = \frac{W}{2} \frac{G_+(0)}{G_+(i)}$ , where the propagators  $G$  are defined below), in what follows we will consider instead  $A$  as the unknown when  $T = 0$ . After a few rearrangements, the relevant equations are now (we still set  $g = \frac{1}{t}$ , although  $t$  does not have to be an integer here)

$$\int_{-\infty}^A n(\theta) \frac{t + e^{2(t-1)(\theta-\theta_B)}}{1 + e^{2(t-1)(\theta-\theta_B)}} d\theta = \frac{U}{2\pi} \quad (24)$$

and

$$I = \int_{-\infty}^A n(\theta) \frac{e^{2(t-1)(\theta-\theta_B)}}{1 + e^{2(t-1)(\theta-\theta_B)}} d\theta. \quad (25)$$

The density  $n(\theta)$  can be computed as a power series in the weak and strong back-scattering limits, giving rise to expansions for the current and the boundary conditions. In the strong back-scattering case one finds

$$I = \frac{G_+(i)}{G_+(0)} \frac{e^A}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{\pi} \Gamma(nt)}{2\Gamma(n)\Gamma\left(\frac{3}{2} + n(t-1)\right)} (e^{A+\Delta-\theta_B})^{2n(t-1)} \quad (26)$$

while the boundary condition is

$$2 \frac{G_+(i)}{G_+(0)} e^A - (t-1) \frac{G_+(i)}{G_+(0)} e^A \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{\pi} \Gamma(nt)}{\Gamma(n)\Gamma\left(\frac{3}{2} + n(t-1)\right)} (e^{A+\Delta-\theta_B})^{2n(t-1)} = U. \quad (27)$$

In the weak back-scattering limit instead, one finds

$$I = \frac{G_+(i)}{G_+(0)} \frac{e^A}{\pi t^2} \left[ t - \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{\pi} \Gamma(n/t)}{2\Gamma(n)\Gamma\left(\frac{3}{2} + n\left(\frac{1}{t} - 1\right)\right)} (e^{A+\Delta-\theta_B})^{2n((1/t)-1)} \right] \quad (28)$$

where

$$\begin{aligned} & \frac{2}{t} \frac{G_+(i)}{G_+(0)} e^A - \frac{1}{t} \left( \frac{1}{t} - 1 \right) \frac{G_+(i)}{G_+(0)} e^A \sum_{n=1}^{\infty} (-1)^{n+1} \\ & \times \frac{\sqrt{\pi} \Gamma(n/t)}{\Gamma(n)\Gamma\left(\frac{3}{2} + n\left(\frac{1}{t} - 1\right)\right)} (e^{A+\Delta-\theta_B})^{2n((1/t)-1)} = U. \end{aligned} \quad (29)$$

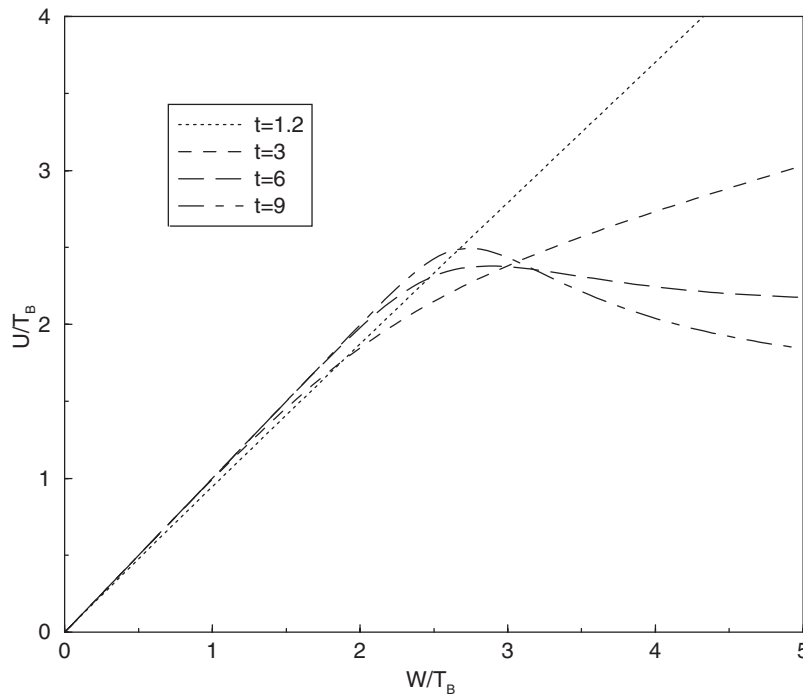
Here we have introduced the notations

$$G_+(\omega) = \sqrt{2\pi t} \frac{\Gamma\left(-i\omega \frac{t}{2(t-1)}\right)}{\Gamma\left(\frac{1}{2} - i\frac{\omega}{2}\right) \Gamma\left(-i\omega \frac{1}{2(t-1)}\right)} e^{-i\omega\Delta} \quad (30)$$

where

$$\Delta = \frac{1}{2} \ln(t-1) - \frac{t}{2(t-1)} \ln t. \quad (31)$$





**Figure 2.** The applied voltage difference  $U/T_B$  as a function of the chemical potential difference between solitons and antisolitons,  $W/T_B$ . Observe the remarkable non-monotonic behaviour that settles in for small enough values of  $g$ . This results in the existence of two possible values of  $W$  for a given  $U$ , and thus in the existence of the loop in the  $I-U$  characteristic.

In terms of the auxiliary variable  $W$ , the strong and weak back-scattering expansions have matching radius of convergence: either one of them is always converging, and both are at the matching value  $\frac{W}{T_B}e^{-\Delta} = 1$ , where the parameter  $T_B'$  is defined as

$$T_B' = 2T_B e^{-\Delta} \frac{G_+(i)}{G_+(0)}. \quad (32)$$

The series can be summed up in the case  $g = \frac{1}{2}$  to give

$$\tan \frac{U - 2\pi I}{2} = \frac{U + 2\pi I}{2}. \quad (33)$$

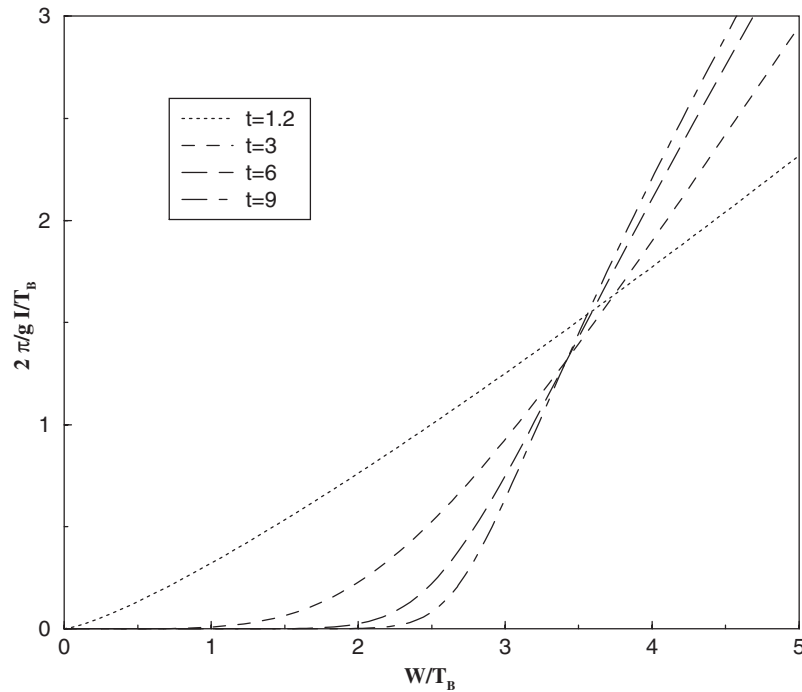
There is a rich physical behaviour hidden in these equations. To investigate it, consider first the behaviour of physical quantities as a function of  $W$ . Curves representing  $U$  and  $\frac{2\pi}{g}I$  as a function of  $W$  for various  $t$  are given in figures 2 and 3.

As  $g \rightarrow 0$ , the current in the strong back-scattering expansion is exactly 0. In the weak back-scattering expansion meanwhile, it becomes

$$\frac{2\pi I}{g} \approx (W^2 - \pi^2 T_B'^2)^{1/2}$$

and hence exhibits a square root singularity at finite value of  $W$  (we note that the latter expression can also be obtained directly from the result (18) by using the uniform asymptotic expansion of Bessel functions for large orders [19]:

$$I_\nu(\mu z) \approx \frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}}{(1+z^2)^{1/4}}$$



**Figure 3.** In contrast, the current  $I$  as a function of  $W$  exhibits, once properly rescaled, a very weak dependence on  $g$ . All curves behave asymptotically as  $W/T_B$  in the weak back-scattering limit.

where  $\eta = \sqrt{1+z^2} + \ln \frac{z}{1+\sqrt{1+z^2}}$ ). When  $t$  is varied, the current evolves from this singular behaviour to the simple characteristics  $I = \frac{W}{4\pi}$  as  $g \rightarrow 1$  (this is easily seen from the integral representations of  $I$  and  $U$  and an artefact of the variable  $T_B$  used throughout, that would have to be rescaled appropriately in that limit to give a non-trivial  $I-U$  relation [14]). At fixed  $g \neq 1$ ,  $I \approx \frac{W}{2\pi}$  at large  $W$ .

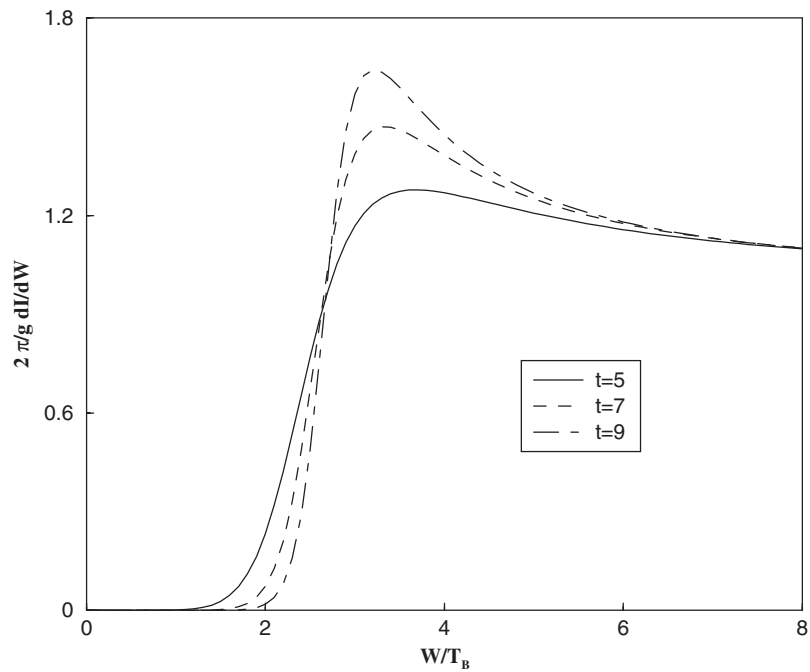
As  $g \rightarrow 0$ ,  $U$  in the strong back-scattering expansion is simply equal to  $W$ , while in the weak back-scattering expansion it is

$$U \approx W - (W^2 - \pi^2 T_B^2)^{1/2}.$$

As  $g \rightarrow 1$  meanwhile,  $U \approx W$ . When  $g$  varies,  $U$  interpolates between these two limiting behaviours and stops having a (local) maximum around  $t \approx 4.83$ .

The fact that  $U$  can *decrease* as  $W$  increases is a direct consequence of the physics in this system. The density on the left,  $\rho_e(-L/2) \propto W$ . An increase in  $W$  increases the left density, but it also increases the right density, since particles being more energetic, more of them go across the impurity.  $U$  is a non-trivial function of the densities on either side of the impurity, as given by (6). For  $g$  large,  $U$  behaves essentially as the sum of the densities in  $\pm L/2$ , and thus increases when  $W$  increases. However, when  $g \rightarrow 0$ ,  $U$  gets dominated by the difference of the densities, and if enough particles go across, it can well decrease when  $W$  increases. This effect is directly related to the fact that the differential conductance  $2\pi \frac{dI}{dW}$  does, for  $g < \frac{1}{2}$ , actually become *larger* than  $g$  for finite values of  $W$  an effect first observed in [5] (see figure 4).

Consider now  $I$  as a function of  $U$ : clearly, the existence of a maximum in the curve  $U(W)$

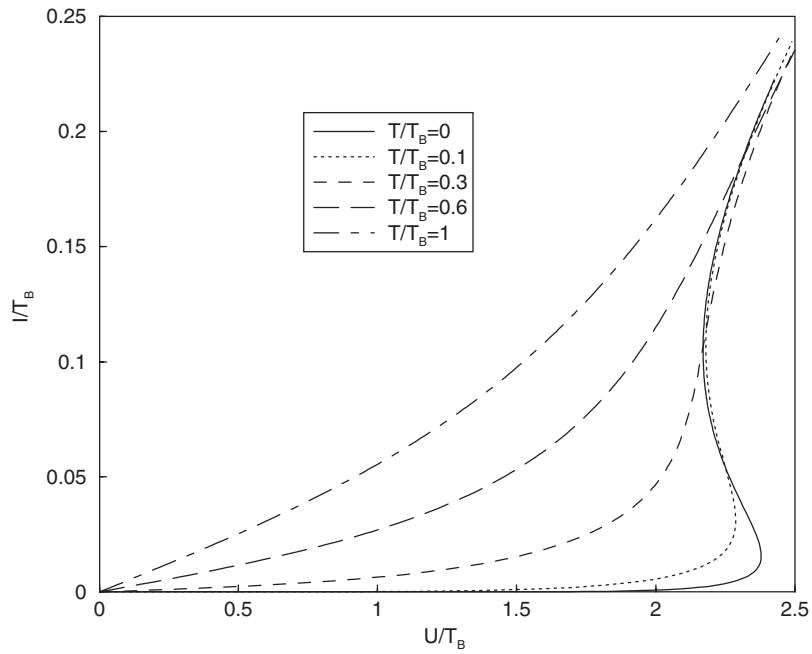


**Figure 4.** The rescaled derivative of the current with respect to  $W$  at  $T = 0$ . Notice the existence of a maximum above the weak back-scattering limit (equal to 1) for  $t \geq 2$ . This peak of differential conductance becomes more and more marked as  $g \rightarrow 0$ .

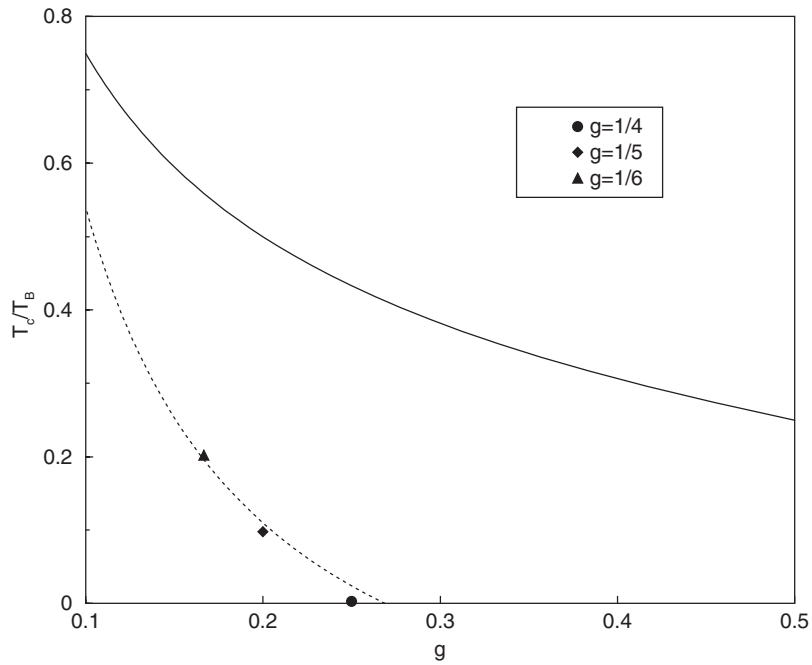
will lead to an S-shaped  $I(U)$ . More precisely, consider first the case  $g \approx 0$ . Suppose we increase  $W$  starting from 0. According to the previous discussion,  $U \approx W - (W^2 - \pi^2 T_B^2)^{1/2}$ , so (as illustrated in figure 2 for slightly larger values of  $g$ )  $U$  first increases up to  $\pi T_B$ , then decreases back to zero.  $W$  being still finite,  $I$  vanishes identically, since it has an overall factor of  $g$ . Going now to the regime where  $W$  becomes infinite,  $U \approx W$  and  $I \approx \frac{W}{2\pi} \approx \frac{U}{2\pi}$ : the system has switched from being a perfect insulator to being a perfect conductor! This is easy to understand in more physical terms: as  $g \rightarrow 0$ , the kinetic term dominates the Lagrangian and one might expect that the impurity is essentially invisible. However, as  $g \rightarrow 0$ , there is the possibility that a charge density wave might form, getting pinned down by an infinitesimal potential, and leading to a perfect insulator [16].

This effect is stable against quantum fluctuations and, for  $g$  approximately smaller than  $g = 0.2$ , a ‘loop’ keeps being observed in the  $I-U$  characteristics. That the current is not a single valued function of  $U$  in the region of small voltages leads to the prediction of hysteresis and bistability in the strongly interacting, out-of-equilibrium regime. Although the present calculation is valid only in the scaling regime, this qualitative aspect should survive beyond it.

The loop is also stable against thermal fluctuations: as is illustrated in figure 5 for the case  $t = 6$ , it only disappears at a finite temperature  $T_c$  which depends on  $g$ . A semiclassical approximation [16] gives  $T_c = T_B \sqrt{\frac{(1-g)}{16g}}$ : this formula is not quite correct for values of  $g \leq 0.2$ , but becomes increasingly good as  $g \rightarrow 0$ . It is quite difficult numerically to determine  $T_c$  with a great accuracy: a reasonable estimate of this curve is given in figure 6.



**Figure 5.** We illustrate in this figure the disappearance of the S shape as the temperature is increased. Clearly, the bistability is stable against thermal fluctuations in a finite range which depends on  $g$ . Here,  $g = 1/6$ .



**Figure 6.** The ‘critical’ temperature  $T_c(g)$  at which bistability disappears. Notice the poor quality of the leading semi-classical approximation (full curve).

#### 4. Duality

For the problem of tunnelling between quantum Hall edges, a striking duality between the weak and strong back-scattering limits was uncovered in [5] at  $T = 0$ , and further generalized to any  $T$  [20]. The meaning of this duality was that, while the Hamiltonian describing the vicinity of the weak back-scattering limit is given by (5), the one describing the vicinity of the strong back-scattering limit can be reduced, as far as the dc current is concerned, to an expression identical with (5), up to the replacement of the coupling  $\lambda$  by a dual coupling  $\lambda_d$ , together with the exchange  $g \rightarrow \frac{1}{g}$ . As a result, a duality relation for the current followed

$$I(\lambda, U, g) = \frac{gU}{2\pi} - gI\left(\lambda_d, gU, \frac{1}{g}\right). \quad (34)$$

Here, the dual coupling  $\lambda_d$  is

$$\lambda_d = \frac{1}{\pi g} \Gamma\left(\frac{1}{g}\right) \left[\frac{g\Gamma(g)}{\pi}\right]^{1/g} \lambda^{-1/g}. \quad (35)$$

The relation (35) follows from keeping the parameter

$$T_B'' \equiv \frac{T_B'}{\sqrt{t}} \quad (36)$$

constant<sup>4</sup> while letting  $g \rightarrow \frac{1}{g}$ , and using the relation (8) between  $T_B$  and the bare coupling in the tunnelling Hamiltonian.

For pedagogical purposes, it is probably wise to explain a little more explicitly what the duality means. Consider thus a hypothetical current defined non-perturbatively by the expression

$$I = \frac{1}{x^2 + g^2}. \quad (37)$$

It obeys the following duality relation

$$I\left(\frac{1}{x}, \frac{1}{g}\right) = g^2 - g^4 I(x, g). \quad (38)$$

Suppose now we did not know the non-perturbative expression, but had only access to the small  $x$  expansion

$$I = \frac{1}{g^2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x^2}{g^2}\right)^n \quad (39)$$

and the large  $x$  one

$$I = \frac{1}{x^2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{g^2}{x^2}\right)^n. \quad (40)$$

The duality (38) could then be deduced from the expansions by, say, starting from the small  $x$  one, setting  $x = \frac{1}{x'}$ ,  $g = \frac{1}{g'}$ , and comparing the new expression with the large  $x$  expansion. What was done in [5] was to find a similar duality, only based on the weak and strong back-scattering expansions (a non-perturbative expression for the current was found much later [21]).

It is interesting to examine what does remain of this duality in the present case. The IR Hamiltonian will behave similarly to the case of tunnelling between quantum Hall edges, since it is entirely determined by the large  $\lambda$  behaviour and has no relation with the way the voltage is

<sup>4</sup> In [5], the duality relation was initially written at constant  $T_B'$ . While the identities in [5] are algebraically correct, it is really  $T_B''$  that has to be kept constant, since the applied voltage is not left invariant in the duality transformation.

taken into account. This means that the parameter  $T_B''$  still has to be kept constant in whatever duality symmetry one is looking for.

There is a quick way to proceed, assuming from [5] the relation (34), which becomes here

$$I(\lambda, W, g) = \frac{gW}{2\pi} - gI\left(\lambda_d, gW, \frac{1}{g}\right). \quad (41)$$

Using this, together with the relation (16), one finds the additional relation

$$U(\lambda, W, g) = U\left(\lambda_d, gW, \frac{1}{g}\right). \quad (42)$$

From this it follows that

$$I(\lambda, U, g) = \frac{U}{2\pi} - I\left(\lambda_d, U, \frac{1}{g}\right). \quad (43)$$

For completeness, we can also give a direct proof of this relation. It is convenient first to put the equations in a more compact form, namely

$$\begin{aligned} \Lambda_s \left[ 1 - (t-1) \sum_{n=1}^{\infty} \alpha_n \Lambda_s^{2n(t-1)} \right] &= u_s \\ i_s(t, u_s) &= \Lambda_s \sum_{n=1}^{\infty} \alpha_n \Lambda_s^{2n(t-1)} \end{aligned} \quad (44)$$

for the strong back-scattering limit and

$$\begin{aligned} \frac{\Lambda_w}{t} \left[ 1 - \left(\frac{1}{t} - 1\right) \sum_{n=1}^{\infty} \beta_n \Lambda_w^{2n((1/t)-1)} \right] &= u_w \\ i_w(t, u_w) &= \frac{1}{t^2} \Lambda_w \left[ t - \sum_{n=1}^{\infty} \beta_n \Lambda_w^{2n((1/t)-1)} \right] \end{aligned} \quad (45)$$

for the weak back-scattering limit. In (44) and (45)

$$\begin{aligned} \alpha_n &= \frac{\sqrt{\pi}}{2} (-1)^{n+1} 2^{n(t-1)} \frac{\Gamma(nt)}{\Gamma(n)\Gamma(\frac{3}{2} + n(t-1))} \\ \beta_n &= \frac{\sqrt{\pi}}{2} (-1)^{n+1} 2^{n((1/t)-1)} \frac{\Gamma(n/t)}{\Gamma(n)\Gamma(\frac{3}{2} + n(\frac{1}{t}-1))} = \alpha_n \left(\frac{1}{t}\right) \\ i_s = i_w &= \sqrt{\frac{2}{t}} \frac{\pi I}{T_B''} \quad u_s = u_w = \sqrt{\frac{2}{t}} \frac{U}{2T_B''}. \end{aligned} \quad (46)$$

To match with our previous notations,  $\Lambda = G_+(i, t) \frac{e^A}{T_B''}$ ; however,  $\Lambda$  in the foregoing equations is determined by the external voltage and no reference to  $e^A$  or  $T_B''$  is necessary in its definition.

It follows from (44) and (45) that

$$i_s(t, u) = \frac{1}{t} \Lambda \left(\frac{1}{t}, tu_w\right) - \frac{1}{t^2} i_w \left(\frac{1}{t}, tu\right) \quad (47)$$

where the parameter  $\Lambda$  is the same in both  $i_s$  and  $i_w$ . Of course, the current is an analytical function of the applied voltage, independent of whether one considers the weak or strong back-scattering expansions, so the labels  $s, w$  can actually be suppressed from the equations. It follows that, going back to physical variables,

$$I(\lambda, U, g) = \frac{1}{2\pi} W \left(\frac{1}{g}, U\right) - gI\left(\lambda_d, U, \frac{1}{g}\right). \quad (48)$$

Now,  $W$  in turn can be expressed in terms of  $U$ ,  $I$ , using the relation (16), reproducing (43).

The relation between the current and the applied voltage is implicit in the foregoing equations. It can, however, be made explicit by elimination of the parameter  $\Lambda$ , and we quote here the lowest orders for completeness. In the weak back-scattering limit one has

$$i = -\frac{1}{t}\beta_1(tu)^{2((1/t)-1)+1} - \left[ \frac{1}{t^3}(t-1)(t-2)\beta_1^2 - \frac{1}{t}\beta_2 \right] (tu)^{4((1/t)-1)+1} + \dots \quad (49)$$

and in the strong back-scattering limit

$$i = \alpha_1 u^{2(t-1)+1} + [(t-1)(2t-1)\alpha_1^2 + \alpha_2] u^{4(t-1)+1} + \dots \quad (50)$$

Meanwhile, the parameter  $\Lambda$  can also be expanded, say in the weak back-scattering limit:

$$\Lambda = \left( \frac{1}{t} - 1 \right) \beta_1(tu)^{2(t-1)+1} + [(t-1)(2t-1)\alpha_1^2 + \alpha_2] u^{4(t-1)+1} + \dots \quad (51)$$

One can directly check the duality relation (47) on these formulae. Notice that, despite the more complex physics, which now involves screening, the exponents of the weak and strong back-scattering expansions are the same as in the fractional quantum Hall case.

Finally, the duality was extended to finite temperatures in [20, 22], meaning that formula (34) holds at finite temperature. Since (16) is still true too, the formula (43) extends to finite temperature as well.

## 5. Conclusions

This paper hopefully solves the tunnelling problem with a proper treatment of the coupling to the reservoirs, hence completing and correcting [5, 15]. We have only treated here the spinless case, but the method extends straightforwardly to the spinfull case, at least when the spin isotropy is not broken, and the problem maps onto a supersymmetric boundary sine-Gordon model [15]

The duality we observed does raise interesting physical questions, in particular concerning the nature of the ‘charges’ that tunnel in the weak back-scattering limit. We hope to get back to this issue with computations of the dc shot noise.

Finally, we stress that the results presented in appendix B provide some of the first direct connections between the traditional Keldysh approach and the one based on integrability and the Landauer–Buttiker formalism. A better understanding of the relations between the two approaches seems one of the major outstanding issues in this field.

## Acknowledgments

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## Appendix A. Semi-classical computations

In studying the classical limit, one usually concentrates on the behaviour of  $\epsilon_j$  for  $j$  finite while  $g \rightarrow 0$ , that is  $t \rightarrow \infty$  [23]. This is not sufficient in the study of transport properties, where the knowledge of  $\epsilon_{\pm}$ , that is pseudoenergies for nodes at the other end of the diagram, are required. The necessary analysis is a bit more complicated then. Firstly, it is convenient to

introduce the new quantity  $Y_j(\theta) \equiv e^{\epsilon_j(\theta)/T}$ , and to recast the TBA system, using the identity  $s(\theta + \frac{i\pi}{2(t-1)}) + s(\theta - \frac{i\pi}{2(t-1)}) = 2\pi\delta(\theta)$ , into

$$Y_j\left(\theta + \frac{i\pi}{2(t-1)}\right) Y_j\left(\theta - \frac{i\pi}{2(t-1)}\right) = [1 + Y_{j+1}(\theta)][1 + Y_{j-1}(\theta)]. \quad (\text{A.1})$$

In the limit where  $g \rightarrow 0$ , we introduce new variables  $s \equiv \frac{i}{t}$ ,  $\alpha \equiv \frac{2\theta}{\pi}$ , and  $e^{-\chi} \equiv \frac{e^\epsilon}{t^2}$ , and expand the left and right hand sides of equation (A.1) to obtain the Liouville equation [24]

$$(\partial_s^2 + \partial_\alpha^2) \chi = 2e^\chi. \quad (\text{A.2})$$

The general solution of this equation that is relevant here is<sup>5</sup>

$$e^{-\chi} = \frac{1}{(2i \sin \pi \rho)^2} \{J_\rho[e^{(\pi/2)(\alpha+is)-\ln(2T)}] J_{-\rho}[e^{(\pi/2)(\alpha-is)-\ln(2T)}] - (\rho \rightarrow -\rho)\}^2 \quad (\text{A.3})$$

where  $J_\rho$  are the usual Bessel functions,  $\rho = \frac{iW}{2\pi T}$ . The freedom in the arguments of the Bessel functions  $\alpha + is \rightarrow \lambda(\alpha - \alpha_0 + i(s - s_0))$  has been resolved by matching with the asymptotic boundary conditions  $\epsilon_j \approx 2 \sin(\frac{j\pi}{2(t-1)})e^\theta$  as  $\theta \rightarrow \infty$ . As for the index of the Bessel functions, it is obtained by matching against the result at low energies:

$$e^{\epsilon_j(-\infty)/T} = \left[ \frac{\sinh(j+1)W/2tT}{\sinh W/2tT} \right]^2 - 1.$$

We can now compute  $\epsilon_{t-2}$  by setting  $s = 1$  in the solution (A.3): one finds

$$e^{-\chi(\alpha,1)} = [J_\rho J_{-\rho}(ie^{(\pi/2)\alpha - \ln(2T)})]^2.$$

It follows that

$$e^{\epsilon_\pm(\theta)/T} = t I_\rho\left(\frac{e^\theta}{2T}\right) I_{-\rho}\left(\frac{e^\theta}{2T}\right). \quad (\text{A.4})$$

The current on the other hand is

$$I = \int_{-\infty}^{\infty} |T_{++}|^2 (\sigma_+ - \sigma_-) d\theta = \frac{T}{2\pi} \int_{-\infty}^{\infty} d\theta \frac{1}{1 + e^{-2(t-1)(\theta - \theta_B)}} \frac{d}{d\theta} \ln \frac{1 + e^{-W/2T} e^{-\epsilon/T}}{1 + e^{W/2T} e^{-\epsilon/T}}.$$

In the limit  $t \rightarrow \infty$ , this becomes then

$$I = \frac{T}{2\pi} 2 \sinh(W/2T) e^{-\epsilon_\pm(\theta_B)/T}. \quad (\text{A.5})$$

Replacing  $\epsilon_\pm$  by his classical expression reproduces then the result (18).

## Appendix B. Low temperature expansion

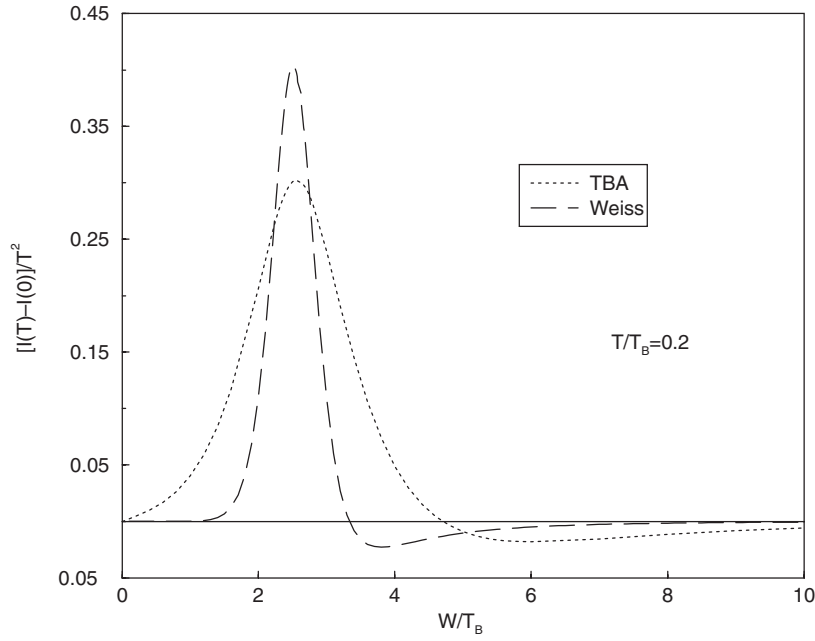
The remarkable relation [25]:

$$I(W, T) = I(W, T = 0) + \frac{\pi^2 T^2 t}{3} \frac{d^2 I}{dW^2}(W, T = 0) \quad (\text{B.1})$$

was initially discovered, following a Keldysh expansion of the left and right hand sides, in the context of dissipative quantum mechanics in [26]. In (B.1),  $W$  is the chemical potential defined in the text—it would coincide with the Hall voltage  $V$  in the context of the fractional quantum Hall effect [5].

<sup>5</sup> This is of the general form of solution  $e^{-\chi} = \frac{(1-A(z)B(\bar{z}))^2}{4\partial A \partial B}$  for the equation  $\partial \bar{\partial} \chi = \frac{1}{2} e^\chi$ , where  $A = \frac{J_{-\rho}}{J_\rho}(e^z)$ ,  $B = \frac{J_\rho}{J_{-\rho}}(e^{\bar{z}})$ .





**Figure B.1.** The broken curve is the order  $T^2$  correction to the non-equilibrium current as estimated by equation (B.1). The dotted curve is the same correction calculated from the TBA at  $T = 0.2$  (it is difficult, for technical reasons, to go below this value with enough accuracy). The two curves are in good qualitative agreement: notice that both of them are below the axis in the weak back-scattering limit. On this figure,  $t = 7$ .

We shall now prove that the current obtained from the TBA does indeed satisfy this relation: as (B.1) involves out-of-equilibrium quantities *and* the temperature, it provides a very non-trivial verification that a Landauer–Buttiker type approach can safely be applied to integrable quasiparticles.

To start, we recall the general expression for the current (15):

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta \frac{d\epsilon}{d\theta} \left( \frac{1}{1 + e^{(\epsilon - W/2)/T}} - \frac{1}{1 + e^{(\epsilon + W/2)/T}} \right) \frac{1}{1 + e^{-2(t-1)(\theta - \theta_B)}} \quad (\text{B.2})$$

where  $\epsilon$  itself is a function of  $T$ . Recall also the value

$$e^{\epsilon(\theta = -\infty, T)/T} = \frac{\sinh(t-1)W/2tT}{\sinh W/2tT}. \quad (\text{B.3})$$

We will only be interested in the terms of order  $T$  and  $T^2$  in the current: we can therefore drop exponentially small contributions, which makes matters considerably simpler. For instance, only the first term in (B.2) contributes and the value of  $\epsilon(-\infty, T)$  coincides at this order with its value for  $T = 0$ ,  $\epsilon(-\infty, 0) \equiv \epsilon_{\min} = \frac{t-2}{2t}W$ .

To proceed, we consider the first term in (B.2) and assume first that  $\epsilon(\theta)$  takes its  $T = 0$  value. The finite  $T$  corrections (we denote them by  $\delta I^{(1)}$ ) then entirely arise from a simple generalization of Sommerfeld's expansion in the case of free electrons. We use here the same notations as in the appendix of [27]. Introducing the function

$$H(\epsilon) = \frac{1}{1 + \left[ \frac{T_B}{e^{\theta(\epsilon)}} \right]^{2(t-1)}} \quad (\text{B.4})$$

we find

$$I = \frac{1}{2\pi} \frac{1}{1 + e^{(\epsilon_{\min} - W/2)/T}} \int_{\epsilon_{\min}}^{\frac{W}{2}} d\epsilon' H(\epsilon') + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^{n-1}}{d\epsilon^{n-1}} H|_{\epsilon=W/2} \int_{\epsilon_{\min}}^{\infty} (\epsilon - W/2)^n \times (-) \frac{d}{d\epsilon} \frac{1}{1 + e^{(\epsilon - W/2)/T}}. \quad (\text{B.5})$$

Since we neglect exponentially small terms, we can neglect the filling fraction in the first prefactor and replace the bound of integration in the integral by  $-\infty$ , using the fact that  $\epsilon_{\min} < \frac{W}{2}$ . It follows similarly that only terms with  $n$  even contribute to the series, and therefore, to leading order,

$$\delta I^{(1)} = \frac{1}{2\pi} \int_{\epsilon_{\min}}^{W/2} d\epsilon' H(\epsilon') + a_1 \frac{T^2}{2\pi} \frac{d}{d\epsilon} H|_{\epsilon=W/2}. \quad (\text{B.6})$$

The first term is nothing but  $I(W, T = 0)$ . As for the second,  $a_1$  is the standard constant of the Sommerfeld expansion

$$a_1 = \int_{-\infty}^{\infty} \frac{\epsilon^2}{2!} \times (-) \frac{d}{d\epsilon} \frac{1}{1 + e^{\epsilon}} d\epsilon = \frac{\pi^2}{6}. \quad (\text{B.7})$$

At the order we are working, we finally obtain

$$\delta I^{(1)}(T) = T^2 \frac{\pi}{12} \left( \frac{d\epsilon}{d\theta} \Big|_{\theta=A} \right)^{-1} \frac{dH}{d\theta} \Big|_{\theta=A} \quad (\text{B.8})$$

where  $A$  is the Fermi momentum introduced in the text. One has, on the other hand,

$$\frac{dH}{d\theta} = \frac{t - 1}{2 \cosh^2(t - 1)(A - \theta_B)}.$$

To proceed, we must also take into account the changes of  $\epsilon$  with temperature in the initial expression of the current. The leading order correction turns out to be of order  $T^2$  then: this gives a second contribution  $\delta I^{(2)}$  to the change of the current and shows that there are no crossed terms to this order.

Neglecting the exponentially small terms as before, the TBA equations for  $\epsilon$  do not need the introduction of other pseudo-energies and are

$$\epsilon(\theta) = e^{\theta} - T \int_{-\infty}^{\infty} \Phi(\theta - \theta') \ln(1 + e^{-(\epsilon(\theta') - W/2)/T}) d\theta'. \quad (\text{B.9})$$

Integrations by part and Sommerfeld expansion give, as in the study of  $I^{(1)}$ , a leading correction going as  $T^2$ . We can thus write  $\epsilon(\theta, T) = \epsilon(\theta, T = 0) + T^2 \delta\epsilon$ , where we find

$$\delta\epsilon(\theta) - \int_{-\infty}^A \phi(\theta - \theta') \delta\epsilon(\theta') d\theta' = -a_1 T^2 \left( \frac{d\epsilon}{d\theta} \Big|_{\theta=A} \right)^{-1} \phi(\theta - A). \quad (\text{B.10})$$

This equation is solved by introducing the operator  $L$  of [22]. Calling the integral operator on the left of (B.10)  $\hat{I} - \hat{K}$  (where  $\hat{I}$  is the identity), one has  $\hat{I} + \hat{L} = \frac{\hat{I}}{\hat{I} - \hat{K}}$ . Using that  $(\hat{I} + \hat{K}) \bullet \phi = L$ , it follows that

$$\delta\epsilon(\theta) = -a_1 T^2 \left( \frac{d\epsilon}{d\theta} \Big|_{\theta=A} \right)^{-1} L(\theta, A). \quad (\text{B.11})$$

Using the value

$$\frac{d\epsilon}{d\theta} \Big|_A = \frac{W}{\sqrt{2t}} \quad (\text{B.12})$$

determined from [22], we find therefore

$$\epsilon(\theta, T) = \epsilon(\theta, T = 0) - T^2 \frac{\pi^2}{3W\sqrt{2/t}} L(\theta, A). \quad (\text{B.13})$$

Of course, the operator  $L$  can be made explicit:

$$L(\theta, \theta') = L(\theta', \theta) = \phi(\theta - \theta') + \int_{-\infty}^A \phi(\theta - \theta'') \phi(\theta'' - \theta') d\theta'' + \dots \quad (\text{B.14})$$

The quantity  $\epsilon$  we use here is related with another quantity  $\epsilon_+^h$  introduced in the main text (23) and studied in great detail in [22], by  $\epsilon = \frac{W}{2} - \epsilon_+^h$ . In the latter reference, the following identity is established:

$$L(\theta, A) = -\sqrt{2t}W \frac{d^2\epsilon}{dW^2}. \quad (\text{B.15})$$

Using this and integrating by parts, we find

$$\delta I^{(2)}(T) = T^2 \frac{\pi}{6W} \sqrt{\frac{t}{2}} \int_{-\infty}^A d\theta L(\theta, A) \frac{dH}{d\theta}. \quad (\text{B.16})$$

So collecting all terms,

$$I(T) = I(T = 0) + T^2 \frac{\pi}{6W} \sqrt{\frac{t}{2}} \left[ \left. \frac{dH}{d\theta} \right|_A + \int_{-\infty}^A L(\theta, A) \frac{dH}{d\theta} d\theta \right]. \quad (\text{B.17})$$

To conclude, we now turn to derivatives of the current with respect to  $W$  at vanishing temperature. The current is usually written as

$$I(T = 0) = \int_{-\infty}^A \rho(\theta) H(\theta) d\theta \quad (\text{B.18})$$

where the density  $\rho$  is given by  $\rho = -\frac{1}{2\pi} \frac{d\epsilon_+^h}{d\theta}$ . Using integration by parts, one has

$$\frac{dI}{dW} = \frac{1}{2\pi} \int_{-\infty}^A \frac{d\epsilon_+^h}{dW} \frac{dH}{d\theta} d\theta. \quad (\text{B.19})$$

Taking another derivative, using (B.12) and (B.15), one finds

$$\frac{d^2 I}{dW^2} = \frac{1}{2\pi W} \sqrt{\frac{1}{2t}} \left[ \left. \frac{dH}{d\theta} \right|_A + \int_{-\infty}^A L(\theta, A) \frac{dH}{d\theta} d\theta \right] \quad (\text{B.20})$$

and thus, comparing with (B.17)

$$I(W, T) = I(W, T = 0) + t \frac{\pi^2 T^2}{3} \frac{d^2 I}{dW^2}(W, T = 0) \quad (\text{B.21})$$

(this, up to exponentially small terms and higher order analytical terms), thus proving the identity.

As commented in the main text and in [5], the differential conductance for  $g < \frac{1}{2}$  is *negative* for large enough  $W/T_B$  (this result does not rely on the Bethe ansatz and is a simple consequence of the non-linear  $I-W$  curve present in the Luttinger liquid). It follows from (B.1) that, for such values of  $g$ , the current in the fractional quantum Hall problem *diminishes* when  $T$  is increased from  $T = 0$ , provided  $W/T_B$  is large enough. This is a rather counterintuitive phenomenon: *a priori*, one expects that, the larger  $T$ , the more energy there is, and therefore the less important the back-scattering should be. Of course, the current depends on more complex details than the overall energy, and it is well possible that  $W$ ,  $T$ , and the non-trivial interactions produce an overall less efficient population of quasiparticles, even though  $T$  is

increased. Notice that the current can also decrease when  $T$  is turned on now at fixed  $U$ , as is clear in figure 5.

To conclude, observe that, using (B.1) together with the duality relation at  $T = 0$ , the same relation is found to hold to order  $T^2$ , in agreement with the fact that the duality relation should actually hold at any temperature.

## References

- [1] Kane C and Fisher M P 1992 *Phys. Rev. B* **46** 15233
- [2] Saminadayar L, Glattli D C, Jin Y and Etienne B 1997 *Phys. Rev. Lett.* **75** 2526  
(Saminadayar L, Glattli D C, Jin Y and Etienne B 1997 *Preprint cond-mat/9706037*)  
de-Picciotto R, Reznikov M, Heiblum M, Umansky V, Bunin G and Mahalu D 1997 *Nature* **389** 162
- [3] Moon K, Yi H, Kane C L, Girvin S M and Fisher M P 1993 *Phys. Rev. Lett.* **71** 4381
- [4] Miliken F P, Umbach C P and Webb R A 1996 *Solid State Commun.* **97** 309
- [5] Fendley P, Ludwig A and Saleur H 1995 *Phys. Rev. B* **52** 8934
- [6] Tans S J, Devoret M H, Dai H, Thess A, Smalley R E, Gerlings L J and Dekker C 1997 *Nature* **386** 474  
Egger R and Gogolin A O 1997 *Phys. Rev. Lett.* **79** 5082  
Kane C L, Balents L and Fisher M P A 1997 *Phys. Rev. Lett.* **79** 5086
- [7] Tarucha S, Honda T and Saku T 1995 *Solid State Commun.* **94** 413  
Yacoby A, Stormer H L, Wingreen N S, Pfeiffer L N, Baldwin K W and West K W 1996 *Phys. Rev. Lett.* **77** 4612
- [8] Maslov D L and Stone M 1995 *Phys. Rev. B* **52** R5539  
Maslov D 1995 *Phys. Rev. B* **52** R14638  
Safi I and Schulz H J 1995 *Phys. Rev. B* **52** R17040  
Furusaki A and Nagaosa N 1996 *Phys. Rev. B* **54** R5239  
Oreg Y and Finkel'stein A M 1996 *Phys. Rev. B* **54** R14625  
Yu Alekseev A, Cheianov V V and Fröhlich J 1996 *Phys. Rev. B* **54** R17320  
Safi I and Schulz H 1997 *Phys. Rev. B* **55** R7331  
Safi I 1998 A dynamic scattering approach for a gated interacting wire *Preprint cond-mat/9807122*
- [9] Egger R and Grabert H 1998 *Phys. Rev. B* **58** 10761  
(Egger R and Grabert H 1998 *Preprint cond-mat/9805268*)
- [10] Landauer R 1957 *IBM J. Res. Dev.* **1** 223  
Landauer R 1970 *Phil. Mag.* **21** 863  
Landauer R 1987 *Z. Phys. B* **68** 217
- [11] Buttiker M 1990 *Phys. Rev. Lett.* **65** 2901
- [12] Egger R, Trauzettel B and Grabert H Landauer transport theory for disordered Luttinger liquids: current-voltage characteristics and non-equilibrium DC noise unpublished
- [13] Ghoshal S and Zamolodchikov A B 1994 *Int. J. Mod. Phys. A* **9** 3841
- [14] Fendley P, Saleur H and Warner N 1994 *Nucl. Phys. B* **430** 577
- [15] Lesage F, Saleur H and Simonetti P 1997 *Phys. Rev. B* **56** 7598
- [16] Egger R, Grabert H, Koutouza A, Saleur H and Siano F 2000 *Phys. Rev. Lett.* **84** 3682
- [17] Callan C G, Klebanov I, Ludwig A W W and Maldacena J 1994 *Nucl. Phys. B* **422** 417
- [18] Fendley P and Saleur H unpublished  
Bazhanov V, Lukyanov S and Zamolodchikov A 1997 *Commun. Math. Phys.* **190** 247
- [19] Abramowitz M and Stegun L *Handbook of Mathematical Functions* (New York: Dover)
- [20] Bazhanov V, Lukyanov S and Zamolodchikov A B 1996 *Commun. Math. Phys.* **177** 381  
Bazhanov V, Lukyanov S and Zamolodchikov A B 1997 *Commun. Math. Phys.* **190** 247  
Bazhanov V, Lukyanov S and Zamolodchikov A B 1997 *Nucl. Phys. B* **489** 487
- [21] Fendley P and Saleur H 1998 *Phys. Rev. Lett.* **81** 2518
- [22] Lesage F and Saleur H 1999 *Nucl. Phys. B* **546** 4540
- [23] Fowler M 1982 *Phys. Rev. B* **26** 2514
- [24] Krichever I, Lipan O, Wiegmann P and Zabrodin A 1997 *Commun. Math. Phys.* **188** 267 and references therein
- [25] Weiss U 1996 *Solid State Commun.* **100** 281
- [26] Weiss U, Sassetti M, Negele T and Wollensak M 1991 *Z. Phys. B* **84** 471
- [27] Ashcroft N and Mermin N D *Solid State Physics* (Sanders College)